Math 3280 Tutorial 4

random variable: X is a real valued function $X: S \to \mathbb{R}$.

1. Probability mass function $\underline{p(a) = P(X=a)}$. X is a discrete r.v.
2. Expected valued $E[X] = \sum_{x: p(x)>0} x \cdot p(x)$
3. $E[g(X)] = \sum_i g(x_i) \cdot p(x_i)$, if $g(x) = ax+b$, $E[aX+b] = aE[X] + b$.
4. $Var[X] = E[(X-E(X))^2] = E[X^2] - (E(X))^2 \geq 0$, $Var(aX+b) = a^2 \cdot Var(X)$.
5. Bernolli r.v: random experiment, $X = \begin{cases} 1 & \text{success} \\ 0 & \text{failure} \end{cases}$

$$p = P(X=1). \quad P(X=0) = 1-p.$$

6. Binomial r.v. (n, p): n independent trials, each results in a success with probability p and failure with probability $(1-p)$. Let X denote the number of success that appear in n trials.

Example ①: X denote a r.v. that takes values -1, 0 and 1 with respective probabilities

$$P(X=-1) = 0.2, \quad P(X=0) = 0.5, \quad P(X=1) = 0.3$$

$E[X]$, $E[X^2]$, $Var(X)$, $E[-2X+1]$, $Var[-2X+1]$.

Solution: $E[X] = \sum_i x_i \cdot p(x_i) = (-1) \times 0.2 + 0 \times 0.5 + 1 \times 0.3 = 0.1$

$E[X^2] = \sum_i x_i^2 \cdot p(x_i) = (-1)^2 \times 0.2 + 0^2 \times 0.5 + 1^2 \times 0.3 = 0.5$

$Var(X) = E[(X-E(X))^2] = \sum_i (x_i - E(X))^2 \cdot p(x_i) = \cdots$

$= E[X^2] - (E[X])^2 = 0.5 - 0.1^2 = 0.49$

$E[-2X+1] = -2 \times E[X] + 1 = -2 \times 0.1 + 1 = 0.8$

$Var[-2X+1] = (-2)^2 \times Var(X) = 4 \times 0.49 = 1.96$.

Example 2. Suppose a biased coin that lands on head with probability p is flipped 10 times. Given that the coin lands on head 6 times, what is the probability that the first 3 outcomes are

① h, t, t (e.g. heads, tails, tails).

② t, h, t.

Solution: X denote the number of heads in the outcomes. X is a binomial r.v. $(10, p)$.

E: the first 3 outcomes are h, t, t.

F: the first 3 - - - - $\boxed{t, h, t}$.

① $P(E \mid X=6) = P(E \cap \{X=6\}) / P(X=6)$

$$P(X=6) = \binom{10}{6} p^6 \cdot (1-p)^4$$

$$P(E \cap \underbrace{\{X=6\}}) =$$

E: h, t, t. (one heads in first 3 outcomes.) 2 tails in first 3 trials

$E \cap \{X=6\}$: 5 heads in the remaining 7 trials.

$$P(E \cap \{X=6\}) = p \cdot (1-p)^2 \cdot \binom{7}{5} p^5 \cdot (1-p)^2 = \binom{7}{5} p^6 \cdot (1-p)^4.$$

$$P(E \mid X=6) = \binom{7}{5} \Big/ \binom{10}{6} = \frac{\frac{7!}{5!\,2!}}{\frac{10!\ 8\times 9\times 10}{6\,6!\ 4!\ 3\times 4}} = \frac{1}{10}.$$

② $$P(F \mid X=6) = \frac{P(F \cap \{X=6\})}{P(X=6)} = \frac{p(1-p)^2 \cdot \binom{7}{5} p^5 (1-p)^2}{\binom{10}{6} \cdot p^6 (1-p)^4} = \frac{1}{10}.$$

Example 3. For a <u>non-negative integer-valued</u> random variable X, we have
$$E[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

Solution: $P(X\geq k)=\sum_{j=k}^{\infty}P(X=j)$

$$\sum_{k=1}^{\infty}P(X\geq k)=\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}P(X=j)$$ ✓

$$=\sum_{j=1}^{\infty}\sum_{k=1}^{j}P(X=j)$$

$$=\sum_{j=1}^{\infty}j\cdot P(X=j)$$

$$=\sum_{j=0}^{\infty}j\cdot P(X=j)$$

$$=E[X]$$

Example 4. Suppose that $P(X=0)=1-P(X=1)$. If we have $E[X]=3\cdot Var(X)$, find $P(X=0)$.

Solution: $p=P(X=0)$.

$$E[X]=0\cdot P(X=0)+1\cdot P(X=1)=1-p$$

$$Var[X]=E[X^2]-(E(X))^2=0^2\cdot P(X=0)+1^2\cdot P(X=1)-(1-p)^2$$

$$=(1-p)-(1-p)^2$$

$$=p(1-p)$$

$$1-p=3\times p\cdot(1-p)\Rightarrow p=\tfrac{1}{3}.$$

$$1=3p \text{ or } 1-p=0 \Rightarrow p=1/3 \text{ or } p=1.$$